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Yangian Realization for Dirac Oscillator

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We investigate the realizations of Yangian algebra for a Dirac oscillator. Applying the representation theory of $Y(sl(2))$ to Dirac oscillator, shift operators for different energy levels for this system are obtained.

KEY WORDS: Yangian; Dirac oscillator; shift operators.

1. INTRODUCTION

In the 1960s, Yang (1967) and Baxter (1982) separately established Quantum Yang–Baxter Equation (For short, QYBE). Since then the investigations on quantum integrable models have been greatly promoted. It is worth mentioning especially that the Yangian and quantum algebra, established by Drinfeld (1985, 1986, 1988) in 1985, offer a cogent mathematical method to study the symmetry of quantum integrable models in Physics. After several decades, a great deal of mathematical ingenuity has gone into solutions and symmetries of quantum integrable models, giving new physical understanding and theoretical results (Ge *et al.*, 1999, 2000; Ba *et al.*, 2001). In connection with the symmetries we are impressed with Yangian algebra. There is a close relationship between many-body problems and Yangian algebra, which describes non-linear quantum space and is related to RTT relation that describes a large number of integrable models.

Dirac oscillator is one kind of Dirac equation which is linear in both momentum and coordinates. In the non-relativistic limit, the equaiton corresponds a three-dimensional isotropic harmonic oscillator with a strong spin–orbit coupling term, hence the name Dirac oscillator (Moshinsky and Szczepaniak, 1989). The physical interpretation of Dirac oscillator is an anomalous chromomagnetic dipole interacting with a particular chromoelectric field. Itô *et al.* (1967) pointed out that the Dirac oscillator can be solved exactly. In 1971, its spectrum was found to present

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usual accidental degeneracies (Cook, 1971). Quesne and Moshinsky (1990) explained the degeneracies of the spectrum by a standard symmetry Lie algebra. In the meanwhile, Delange (1991) gave the shift operators, for the Hamiltonian of the equation using operator methods to Dirac equation of Moshinsky and Szczepaniak (1989). As we know, Yangian cannot only describe the symmetry properties, but also give shift operators for energy. Actually, Yangian is much larger than Lie algebra; the shift operators given by Yangian are quite different from that offered by Lie algebra. The difference can be shown in the following sense; the generators of Lie algebra only shift the spectrum in the same energy level, however, Yangian could connect states of different energy power. It seems that we can obtain the shift operators for Dirac oscillator by the method of Yangian.

In this paper, we would like to find the shift operators based on Yangian algebra. In Section 2, we review some properties of Dirac oscillator and give Yangian realization for the system. We use Yangian generators to obtain the shift operators for energy in Section 3. Then we compare our shift operators with the results of Delange (1991). We find another Yangian realization for Dirac oscillator in Section 4, based on the symmetry Lie algebra for Dirac oscillaotr. Consequently, one set of shift operators other than the above ones are given. Using these two sets of shift operators, we are able to transfer the states freely. In Section 5, we summarize our results.

2. DIRAC OSCILLATOR AND ITS YANGIAN REALIZATION

Dirac oscillator is a Dirac equation

$$i\hbar\frac{\partial}{\partial t}|\varphi\rangle = H|\varphi\rangle \tag{1}$$

with the Hamiltonian specified as the following:

$$H = c\vec{\alpha}\cdot(\vec{p} - im_0\omega\beta\vec{r}) + m_0c^2\beta \tag{2}$$

where $\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$ and $\vec{\sigma}$ are Pauli operators, $\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, m_0 is rest mass and ω is the frequency of the oscillator. The solution of the equation can be written as $|\varphi\rangle = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} e^{-i\frac{E}{\hbar}t}$ with the following equations satisfied in the non-relativistic limit,

$$H_1\varphi_1 = \lambda_{\varphi_1}$$

$$H_2\varphi_2 = \lambda_{\varphi_2}$$

$$H_1 = \frac{1}{2m_0\omega\hbar}p^2 + \frac{m_0\omega}{2\hbar}r^2 - \frac{2\vec{L}\cdot\vec{S}}{\hbar^2} - \frac{3}{2}$$

$$H_2 = \frac{1}{2m_0\omega\hbar}p^2 + \frac{m_0\omega}{2\hbar}r^2 + \frac{2\vec{L}\cdot\vec{S}}{\hbar^2} + \frac{3}{2}$$
$$\lambda = \frac{E^2 - m_0^2c^4}{2m_0\omega\hbar c^2} \tag{3}$$

where φ_1 and φ_2 are time-independent components. Such Hamiltonians H_1 and H_2 respresent standard harmonic oscillator plus a very strong spin-orbit coupling and consequently, the system is referred as Dirac oscillator. The energy spectrum is given by

$$\lambda_{Nij} = N - \left[j(j+1) - l(l+1) - \frac{3}{4} \right] \tag{4}$$

Here, N denotes the eigenvalue of $\hat{N} = \frac{1}{2m_0\omega\hbar}p^2 + \frac{m_0\omega}{2\hbar}r^2 - \frac{3}{2}$ and runs over $0, 1, 2, \ldots, l$ and j are the orbital and total angular momentum quantum numbers respectively. The Hamiltonian Equation (2) commutes with the total angular momentum operator $\vec{M} = \vec{L} + \vec{S}$ and the operator $K = \begin{pmatrix} \mathcal{K} & 0 \\ 0 & -\mathcal{K} \end{pmatrix}$ with $\mathcal{K} = \frac{2\vec{L}\cdot\vec{S}}{\hbar^2} + 1$.

A Yangian is formed by a set $\{\vec{I}, \vec{J}\}$ obeying the commutation relations Drinfeld (1985),

$$[I_\alpha, I_\beta] = c_{\alpha\beta\gamma} I_\gamma, \quad [I_\alpha, J_\beta] = c_{\alpha\beta\gamma} J_\gamma$$
$$[J_\alpha, [J_\beta, I_\gamma]] - [I_\alpha, [J_\beta, J_\gamma]] = h^2 a_{\alpha\beta\gamma\lambda\mu\nu} \sum_{\lambda\neq\mu\neq\nu} I_\lambda I_\mu I_\nu$$
$$[[[J_\alpha, J_\beta], [I_\sigma, J_\tau]] + [[J_\sigma, J_\tau], [I_\alpha, J_\beta]]] = h^2(a_{\alpha\beta\gamma\lambda\mu\nu}c_{\sigma\tau\gamma} - a_{\sigma\tau\gamma\lambda\mu\nu}c_{\alpha\beta\gamma}) \times \sum_{\lambda\neq\mu\neq\nu} I_\lambda I_\mu I_\nu \tag{5}$$

where h is an arbitrary constant and $a_{\alpha\beta\gamma\lambda\mu\nu} = \frac{1}{4!}c_{\alpha\lambda\sigma}c_{\beta\mu\tau}c_{\gamma\nu\rho}c_{\sigma\tau\rho}$, $(\alpha, \beta, \gamma, \lambda, \mu, \nu, \sigma, \rho, \tau = 1, 2, 3)$. With $I_\pm = I_1 \pm iI_2$ and $J_\pm = J_1 \pm iJ_2$. Equation (5) can be written as

$$[I_3, I_\pm] = \pm I_\pm, \quad [I_+, I_-] = 2I_3$$
$$[I_3, J_\pm] = [J_3, I_\pm] = \pm J_\pm, \quad [I_+, J_-] = [J_+, I_-] = 2J_3$$
$$[I_\pm, [J_3, J_\pm]] = \frac{1}{4!}h^2 I_\pm(I_3J_\pm - J_3I_\pm) \tag{6}$$

There are many physical realizations of $Y(sl(2)) = Y(\vec{I}, \vec{J})$ satisfying Equation (6). The algebraic meaning of $Y(sl(2))$ is clear that it contains $sl(2)$ as shown by the first line of Equation (6) as a subalgebra. If we have two operators $\vec{E}$ and $\vec{B}$ of

E(3) which satisfy

$$[E_\alpha, E_\beta] = i\epsilon_{\alpha\beta\gamma} E_\gamma$$
$$[E_\alpha, E_\beta] = i\epsilon_{\alpha\beta\gamma} B_\gamma, \quad \alpha, \beta, \gamma = 1, 2, 3$$
$$[B_\alpha, B_\beta] = 0 \tag{7}$$

then it is easy to verify that the following $\{\vec{I}, \vec{J}\}$

$$\vec{I} = \vec{E}, \quad \vec{J} = \vec{I}^2 \vec{B} \tag{8}$$

form a Yangian algebra and for a Dirac oscillator, we obtain

$$\vec{I} = \vec{M} = \vec{L} + \vec{S}, \quad \vec{J} = \vec{M}^2(\vec{p} + c\vec{r}) \tag{9}$$

where c is an arbitrary constant. Such definations satisfy *Y(sl(2))* algebra, namely we get one kind of realization of Yangian for Dirac oscillator.

3. SHIFT OPERATORS FOR DIRAC OSCILLATOR

If we have such a relation

$$[H, f(\vec{J})] = bf(\vec{J}) \tag{10}$$

where $f(\vec{J})$ is a function of $\vec{J}$ and b is an arbitrary constant, then $f(\vec{J})$ can act as shift operators for the Hamiltonian.

For $c = im_0\omega$, we have

$$J^2 = (\vec{p}_+ im_0\omega\vec{r})^2(M^4 + 2M^2)$$
$$J^{\dagger 2} = (\vec{p}_- im_0\omega\vec{r})^2(M^4 + 2M^2) \tag{11}$$

where $\vec{J}^\dagger$ is conjugate operator of $\vec{J}$. After calculations, we have the commutation relations between H_1, H_2, and J^2, $J^{\dagger 2}$

$$[H_1, J^2] = 2J^2, \quad [H_2, J^2] = 2J^2$$
$$[H_1, J^{\dagger 2}] = -2J^{\dagger 2}, \quad [H_2, J^{\dagger 2}] = -2J^{\dagger 2} \tag{12}$$

Thus, these two operators J^2 and $J^{\dagger 2}$ are the ones acting as the shift operators for H_1 and H_2. To be concrete, J^2 is the operator which adds 2 to eigenvalue λ, $J^{\dagger 2}$ is the operator which subtracts 2 from eigenvalue λ. By further calculations, we have

$$[L^2, J^2] = 0, \quad [L^2, J^{\dagger 2}] = 0$$
$$[M^2, J^2] = 0, \quad [M^2, J^{\dagger 2}] = 0$$
$$[M_3, J^2] = 0, \quad [M_3, J^{\dagger 2}] = 0 \tag{13}$$

which means that the two shift operators change the eigenvalue of Hamiltonian by changing the value of N, not affecting the quantum numbers l, j, and m.

We can rewrite the expressions of the shift operators in terms of H_1, H_2, and $\mathcal{K}$:

$$\begin{aligned} J^2 &= 2m_0\omega\hbar\left[\frac{i}{2\hbar}(\vec{r}\cdot\vec{p}+\vec{p}\cdot\vec{r}) - \frac{m_0\omega r^2}{\hbar}H_1 + \mathcal{K} + \frac{1}{2}\right](M^4+2M^2) \\ &= 2m_0\omega\hbar\left[\frac{i}{2\hbar}(\vec{r}\cdot\vec{p}+\vec{p}\cdot\vec{r}) - \frac{m_0\omega r^2}{\hbar}H_2 - \mathcal{K} - \frac{1}{2}\right](M^4+2M^2) \\ J^{\dagger 2} &= 2m_0\omega\hbar\left[\frac{-i}{2\hbar}(\vec{r}\cdot\vec{p}+\vec{p}\cdot\vec{r}) - \frac{m_0\omega r^2}{\hbar}H_1 + \mathcal{K} + \frac{1}{2}\right](M^4+2M^2) \\ &= 2m_0\omega\hbar\left[\frac{-i}{2\hbar}(\vec{r}\cdot\vec{p}+\vec{p}\cdot\vec{r}) - \frac{m_0\omega r^2}{\hbar}H_2 - \mathcal{K} - \frac{1}{2}\right](M^4+2M^2) \end{aligned} \tag{14}$$

Suppose we perform J^2 and $J^{\dagger 2}$ on φ_1 and φ_2 respectively, we have

$$\begin{aligned} J^2\varphi_1 &= 2m_0\omega\hbar[j^2(j+1)^2+2j(j+1)]\left[\frac{i}{2\hbar}(\vec{r}\cdot\vec{p}+\vec{p}\cdot\vec{r}) \right. \\ &\quad \left. - \frac{m_0\omega r^2}{\hbar} + \lambda + k + \frac{1}{2}\right]\varphi_1 \\ J^2\varphi_2 &= 2m_0\omega\hbar[j^2(j+1)^2+2j(j+1)]\left[\frac{i}{2\hbar}(\vec{r}\cdot\vec{p}+\vec{p}\cdot\vec{r}) \right. \\ &\quad \left. - \frac{m_0\omega r^2}{\hbar} + \lambda + k + \frac{1}{2}\right]\varphi_2 \\ J^{\dagger 2}\varphi_1 &= 2m_0\omega\hbar[j^2(j+1)^2+2j(j+1)]\left[\frac{-i}{2\hbar}(\vec{r}\cdot\vec{p}+\vec{p}\cdot\vec{r}) \right. \\ &\quad \left. - \frac{m_0\omega r^2}{\hbar} + \lambda + k + \frac{1}{2}\right]\varphi_1 \\ J^{\dagger 2}\varphi_2 &= 2m_0\omega\hbar[j^2(j+1)^2+2j(j+1)]\left[\frac{-i}{2\hbar}(\vec{r}\cdot\vec{p}+\vec{p}\cdot\vec{r}) \right. \\ &\quad \left. - \frac{m_0\omega r^2}{\hbar} + \lambda + k + \frac{1}{2}\right]\varphi_2 \end{aligned} \tag{15}$$

where $j(j+1)$ is the eigenvalue of M^2 and k denotes the Dirac quantum number which is the eigenvalue of $\mathcal{K}$.

Let us mention the results in Delange (1991)

$$Q_\lambda^+\varphi_1 = \left[\frac{i}{2\hbar}(\vec{r}\cdot\vec{p}+\vec{p}\cdot\vec{r}) - \frac{m_0\omega r^2}{\hbar} + \lambda + k + \frac{1}{2}\right]\varphi_1$$

$$Q_\lambda^+\varphi_2 = \left[\frac{i}{2\hbar}(\vec{r}\cdot\vec{p}+\vec{p}\cdot\vec{r}) - \frac{m_0\omega r^2}{\hbar} + \lambda + k - \frac{1}{2}\right]\varphi_2$$

$$Q_\lambda^-\varphi_1 = \left[\frac{-i}{2\hbar}(\vec{r}\cdot\vec{p}+\vec{p}\cdot\vec{r}) - \frac{m_0\omega r^2}{\hbar} + \lambda + k + \frac{1}{2}\right]\varphi_1$$

$$Q_\lambda^-\varphi_2 = \left[\frac{-i}{2\hbar}(\vec{r}\cdot\vec{p}+\vec{p}\cdot\vec{r}) - \frac{m_0\omega r^2}{\hbar} + \lambda + k - \frac{1}{2}\right]\varphi_2 \tag{16}$$

$Q_\lambda^\pm$ are shift operators for energy, $Q_\lambda^\pm|\varphi_{1,2}\rangle \propto |\varphi_{1,2}\pm 2\rangle$. Compare Equation (15) with Equation (16), we have

$$Q_\lambda^+ = \frac{1}{2m_0\omega\hbar[j^2(j+1)^2+2j(j+1)]}J^2$$

$$Q_\lambda^- = \frac{1}{2m_0\omega\hbar[j^2(j+1)^2+2j(j+1)]}J^{\dagger 2} \tag{17}$$

It was shown in Delange (1991) that $Q_\lambda^\pm$ are scalar operators with respect to orbit angular momentum $\vec{L}$ and total angular momentum $\vec{M}$. They do not affect the quantum numbers j, l, m, k. Obviously, there are good correspondences between Q_λ^+ and J^2, Q_λ^- and $J^{\dagger 2}$. The meaning of the letter is to reconsider the problem of shift operators from the view of Yangian, giving a new explanation of $Q_\lambda^\pm$ which are rooted in Yangian algebra. It is an interesting result. Usually, the generators of Yangian, $I_\pm$, I_3, and $J_\pm$, J_3, combined together to act as shift operators for energy. Now we obtain the square of $\vec{J}$ which can also be used as shift operators.

4. ANOTHER YANGIAN REALIZATION

In their 1990 paper, Quesne and Moshinsky (1990) proved that the symmetry Lie algebra of Dirac oscillator is SO(4) ⊕ SO(3, 1). They restricted their paper to the eqation $H_1\varphi_1 = \lambda\varphi_1$. For convenience, the Hilbert space $\mathcal{H}$ spanned by the eigenfunction φ_1 is divided into two subspaces $\mathcal{H}^+$ and $\mathcal{H}^-$, containing eigenfunctions with $l = j + \frac{1}{2}$ and $l = j - \frac{1}{2}$ respectively. SO(4) accounts for the finite-degenerate levels in $\mathcal{H}^+$, while SO(3, 1) accounts for the infinite-degenerate levels in $\mathcal{H}^-$. Turn our attention to the subspace $\mathcal{H}^+$, the generators of SO(4) are

$$\vec{D} = P^{(+)}\vec{M}P^{(+)}$$

$$\vec{A} = P^{(+)}\frac{1}{4}[(\vec{M}+2)^{-1}(H_1+2\vec{M}+2)^{1/2}\vec{F} + H_1\vec{M} + \vec{G}(H_1+2\vec{M}+2)^{1/2}(\vec{M}+2)^{-1}]P^{(+)} \tag{18}$$

Here, $P^{(+)}$ is the projection operator on $\mathcal{H}^+$, $\vec{M} = [M^2+\frac{1}{4}]^{1/2} - \frac{1}{2}$, the components F_q and G_q $(q = -1, 0, 1)$ of $\vec{F}$ and $\vec{G}$ are

$$F_q = \eta_q(\vec{N} - \vec{L}) - (\vec{\eta}\cdot\vec{\eta})\xi_q,\ \ G_q = (-1)^q(F_{-q})^\dagger \tag{19}$$

where $\vec{L} = [L^2+\frac{1}{4}]^{1/2} - \frac{1}{2}$, η_q and ξ_q are creation and annhilation operators for Dirac oscillator.

Introduce two vectors

$$\vec{I}' = \vec{D}, \qquad \vec{J}' = \frac{h}{4i}\vec{D}\times\vec{A} + F\vec{D} \tag{20}$$

where h is an arbitrary constant and $[F, \vec{D}] = [F, \vec{A}] = 0$. It has been shown that such defined $\{\vec{I}', \vec{J}'\}$ also form a $Y(sl(2))$ (Ge and Xue, 1999), namely, this is another realization of Yangian for Dirac oscillator. For such kind of Yangian realization, one set of shift operators for the Hamiltonian were given (Ge and Xue, 1999)

$$\begin{aligned} &[I'^2, O_\alpha^\epsilon] = (f_\epsilon + 4)O_\alpha^\epsilon, \qquad \alpha = \pm, 3 \\ &\left[I_3', O_\pm^\epsilon\right] = \pm O_\pm^\epsilon, \qquad \left[I_3', O_3^\epsilon\right] = 0, \qquad \epsilon = \pm \end{aligned} \tag{21}$$

where

$$\begin{aligned} O_3^\epsilon(f) &= I_+'J_-' - I_-'J_+' + f_\epsilon J_3' - \frac{4}{4+f_\epsilon}(\vec{I}'\cdot\vec{J}')I_3' \\ O_\pm^\epsilon(f) &= \mp 2(I_\pm'J_3' - I_3'J_\pm') + f_\epsilon J_\pm' - \frac{4}{4+f_\epsilon}(\vec{I}'\cdot\vec{J}')I_\pm' \end{aligned} \tag{22}$$

when $f_+ = 2(j-1)$, $f_- = -2(j+2)$, O_α^ϵ shift the quantum number j by 1. And O_3^ϵ and $O_\pm^\epsilon$ differ in the sense that O_3^ϵ do not change the eigenvalue of I_3', namely, m, while $O_\pm^\epsilon$ are still the shift operators for M_3 in the mean time. Because the generators $\vec{I}'$ and $\vec{J}'$ commute with the Hamiltonian, we have

$$\left[H_1, O_\alpha^\epsilon\right] = 0 \tag{23}$$

That is to say that O_α^ϵ are shift operators among different quantum states in the same energy level.

In Fig. 1, the actions of J^2, $J^{\dagger 2}$ and $O_{3,\pm}^{\pm}$ have been shown. Here we restrict our attention to $\mathcal{H}^+$. The similiar results can also be given in the other subspace $\mathcal{H}^-$. One thing different is that we should use $SO(3,1)$ generators to get a new set of $\{\vec{I}, \vec{J}\}$ and consequently, new shift operators $O_{3,\pm}^{\pm}$.

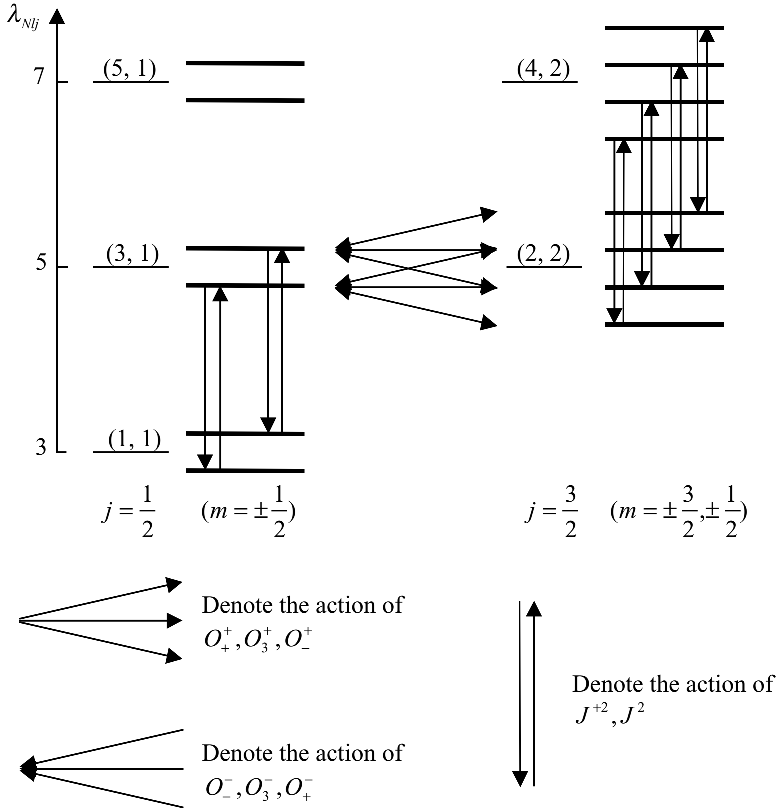

Fig. 1. Shift operators J^2, $J^{\dagger 2}$ and $O^{\pm}_{3,\pm}$ for the Hamiltonian.

Let us mention that there is a kind of realization for Dirac oscillator other than the above two kinds (Ge and Xue, 1999). Choose

$$\vec{I}'' = \vec{L} + \vec{S}, \qquad \vec{J}'' = \vec{L} \times \vec{S} \tag{24}$$

and hence $O_3''^{\pm}$ and $O_\pm''^{\pm}$ are given as

$$\begin{aligned} O_3''^{\pm} &= I_+'' J_-'' - I_-'' J_+'' + f_\pm'' J_3'' - \frac{4}{4 + f_\pm''}(\vec{I}'' \cdot \vec{J}'') I_3'' \\ O_\pm''^{\pm} &= \mp 2(I_\pm'' J_3'' - I_3'' J_\pm'') + f_\pm'' J_\pm'' - \frac{4}{4 + f_\pm''}(\vec{I}'' \cdot \vec{J}'') I_\pm'' \end{aligned} \tag{25}$$

In this case, $\vec{I}'' \cdot \vec{J}'' = 0$. When $f_+'' = 2(j-1)$, $f_-'' = -2(j+2)$, $O''^{\pm}_{3,\pm}$ change the quantum number j to $j+1$. The transfer of states in different subspaces $\mathcal{H}^+$

and $\mathcal{H}^-$ can be realized by $O''^{\pm}_{3,\pm}$. Thus, together with these shift operators, it is possible to connect the states in the whole Hilbert space.

5. CONCLUSION

In conclusion, we start from the theory of Yangian, investigate the Yangian realizations for Dirac oscillator. Based on the generators of Yangian, we obtain shift operators for the Hamiltonian of Dirac oscillator. By two different sets of $\{\vec{I}, \vec{J}\}$, we derive shift operators for different energy levels, J^2, $J^{\dagger 2}$, and the ones for the same energy level $O^{\pm}_{3,\pm}$, respectively. In the former case, the results we got are perfectly in accordance with the shift operators (Delange, 1991). Expressing $Q^{\pm}_{\lambda}$ in terms of Yangian generators, we reconsider the problem of shift operators from the view of Yangian, give a new explanation of $Q^{\pm}_{\lambda}$ which are rooted in Yangian algebra. Combining these two sets of shift operators, we enable the transfer among the states both in the same and different energy levels possible.

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